

Attainability of maximum work and the reversible efficiency from minimally nonlinear irreversible heat engines

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We use the general formulation of irreversible thermodynamics and study the minimally nonlinear irreversible model for heat engines operating between a time varying hot heat source of finite size and a cold heat reservoir of infinite size. We explicitly calculate the condition for obtaining optimized work output for this model once the system reaches the final thermal equilibrium state with that of the cold heat reservoir. We find that our condition resembles with the generalized condition to achieve an optimized work output for generalized irreversible heat engines in the nonlinear regime [Y. Wang, Phys. Rev. E **93**, 012120 (2016)]. We also find that the optimized efficiency obtained by this minimally nonlinear irreversible heat engine can reach the reversible efficiency under the tight coupling condition in which there is no heat leakage between the system and the reservoirs. Under this condition, we find that the reversible efficiency is obtain for any finite time interval with arbitrary power. We also calculate the efficiency at maximum power from the minimally nonlinear irreversible heat engine under the non-tight coupling condition. We find that the efficiency at maximum power is equal to the half of the reversible efficiency and the corresponding maximum work is half of the exergy for a specific choice of the heat leakage term. Our result matches exactly with the efficiency and the work at maximum power obtained in Ref. [Y. Izumida and K. Okuda, Phys. Rev. Lett. **112**, 180603 (2014)] for the exergy study of linear irreversible heat engines under the tight coupling condition. Our study also shows that the efficiency and the work at maximum power obtained from the linear irreversible heat engines under the tight coupling is a special case of the efficiency at maximum power obtained from the minimally nonlinear irreversible heat engine under the non-tight coupling condition.

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I. INTRODUCTION

The theory of irreversible thermodynamics [1–5] nowadays attracts more interest towards the formulation of a new theoretical framework as well as experimental studies of the biological systems and bio-inspired artificial nanosystems [6, 7]. Most of these systems are highly nonlinear and working under the general principle of a heat engine operating in nonequilibrium conditions [8]. A heat engine is a thermodynamic system operating between two heat reservoirs which consumes heat Q_h from the hot heat source at a given temperature T_h and converts part of it in to useful work W and the remaining heat Q_c is delivered in to the cold heat reservoir at a given temperature T_c .

Traditional studies of heat engine are based on the reversible thermodynamics formulation of a linear system operating between the hot and cold reservoirs of infinite size. For an irreversible thermodynamics, most of the studies on heat engine are formulated for linear system operating between the hot and cold heat source of infinite size [9–11]. These studies are focused mainly on obtaining the efficiency $\eta = \frac{W}{Q_h}$, at maximum power in a finite time and its universality behavior $\eta_U \equiv \eta_C/2 + \eta_C^2/8 + O(\eta^3)$, [12–15] where $\eta_C = 1 - T_c/T_h$ is the Carnot efficiency of the reversible heat engine with zero power. The Carnot efficiency, also called as the reversible efficiency, is the maximum efficiency that can

be obtained in the quasi static process taking an infinite time for completion.

For an optimized thermal engine in the endoreversible limit, the efficiency at maximum power is given by $\eta_{CA} = 1 - \sqrt{1 - \eta_C}$ [16, 17] usually called as the Curzon-Ahlborn efficiency. When the temperature difference between two reservoirs are small, the Taylor expansion of η_{CA} gives η_U [18, 19] which is bounded below the Carnot efficiency of the reversible heat engines. It has been shown that, the efficiency at maximum power does not show universality behavior even in the linear response region of certain systems [8]. Since the efficiency of reasonably larger values obtained by the practical heat engines are not working in the regime of maximum power output [20], a recent study showed that universal bounds on efficiency can also be derived for an arbitrary power [21].

The general theory of linear irreversible heat engines working between a finite sized hot heat source and an infinite sized cold reservoir has been formulated recently [22]. This formulation was based on the extraction of maximum work called Exergy [23] obtained from the finite sized hot heat source of time dependent temperature T until the system reaches the final equilibrium state of cold reservoir. More general formulation of optimized maximum work output and the universal feature of the efficiency at maximum power for the irreversible heat engines operated between finite sized heat reservoirs beyond linear regime has been studied very recently [24].

It has been shown that the reversible efficiency can never be reached at finite power for linear irreversible systems [25]. This may raise the question whether it can be reachable for a nonlinear irreversible system at finite power [24]. In order to answer this question we have taken the minimally nonlinear irreversible thermodynamic model [26, 27] in our study.

Even though the minimally nonlinear model can be used to obtain various results derived earlier [28, 29] this model has been criticized in Ref.[30] since the dissipation term present explicitly in this model appears naturally in the linear irreversible Thermodynamic models. Apart from finding the condition for obtaining the reversible efficiency, we try to overcome this criticism by considering this model in exergy calculation. Although this model has been studied partially in Ref. [24] by using perturbation method, we use this model in our exergy study and explicitly calculate the condition to obtain the optimized work and the maximum efficiency without using any approximation methods. We find that the condition derived resembles with the Eq.(26) of Ref. [24]. This may not be obtained in the linear system even for the tight coupling condition in which there is no heat leakage between the system and the reservoirs [22, 31].

This paper has been organized as follows. In section 2, we introduce the minimally nonlinear irreversible model for exergy calculation. In section 3, we incorporate thermodynamical optimization procedure and calculate the optimized efficiency under the tight coupling condition. We also calculate the efficiency at maximum power under the non-tight coupling condition in section 4 and finally conclude with the main results.

II. MINIMALLY NONLINEAR IRREVERSIBLE MODEL

Incorporating the Onsager relation in the study of heat engines [10, 32], a minimal model for a nonlinear heat engine has been introduced by Izumida and Okuda [26, 27] which is given by

$$J_1 = L_{11}X_1 + L_{12}X_2 \quad (1)$$

$$J_2 = L_{21}X_1 + L_{22}X_2 - r_h J_1^2, \quad (2)$$

where J_i is thermodynamic flux and X_i is its conjugate thermodynamic force defined as

$$J_1 \equiv \dot{x}, \quad (3)$$

$$X_1 \equiv F/T, \quad (4)$$

$$J_2 \equiv \dot{Q}_h, \quad (5)$$

$$X_2 \equiv \frac{1}{T_c} - \frac{1}{T}, \quad (6)$$

and L'_{ij} s are the Onsager coefficients with the reciprocity relation $L_{12} = L_{21}$ [22]. For the nonnegativity of the entropy production rate [22, 26], the possible values of L_{ij}

are restricted as $L_{11} \geq 0$, $L_{22} \geq 0$ and $L_{11}L_{22} - L_{12}L_{21} \geq 0$. The non linear term $r_h J_1^2$ was introduced to account for the dissipation effect with $r_h > 0$ in the Onsager relation [10, 26]. In the above equations, F denotes the time independent external generalized force and \dot{x} denotes the time derivative of its conjugate variable x . The terms \dot{Q}_h , \dot{Q}_c and \dot{W} are the time derivatives of Q_h , Q_c and W respectively.

In order to find out the optimized efficiency, we use the extended Onsager relation as described in Eqs.(1) and (2) and study the exergy of nonlinear irreversible heat engines operating between a time varying hot heat source of finite size and a cold heat reservoir of infinite size. The system finally reaches the thermal equilibrium state with a uniform temperature of the cold heat reservoir. We calculate the condition for obtaining the optimized efficiency as follows.

The time varying hot heat source, initially in equilibrium at temperature T_h , is assumed to be always in equilibrium for any other temperature T at later time. The heat capacity at constant volume at any temperature is $C_v = C_v(T)$ and the initial internal energy and entropy are U_h and S_h respectively. When the hot heat source approaches the final temperature of the cold reservoir in a time interval 0 to τ one can calculate the total work extracted by the heat engine as $W = \int dW = \int \eta^T dQ_h$ where dQ_h is the infinitesimal heat that can be transformed into the infinitesimal work dW with the efficiency η^T at each T . This work can be bounded by the Carnot efficiency $\eta_C^T = 1 - T_c/T$ at each T which is given by [22]

$$\begin{aligned} W &\leq \int \eta_C^T dQ_h = - \int_{T_h}^{T_c} \eta_C^T C_v dT \\ &= (U_h - U_c) - T_c(S_h - S_c) \\ &\equiv E, \end{aligned} \quad (7)$$

where

$$U_h - U_c \equiv \int_{T_c}^{T_h} C_v dT, \quad (8)$$

$$S_h - S_c \equiv \int_{T_c}^{T_h} \frac{C_v}{T} dT, \quad (9)$$

U_c and S_c are respectively the internal energy and entropy of the final equilibrium state reached by the engine at $T = T_c$ and E is the maximum work called as the exergy. The corresponding efficiency $\eta = W/Q_h = W/(U_h - U_c)$ is bounded below the maximum value as [22]

$$\begin{aligned} \eta &\leq \frac{E}{U_h - U_c} = 1 - \frac{T_c(S_h - S_c)}{U_h - U_c} \\ &\equiv \eta_{max}, \end{aligned} \quad (10)$$

where η_{max} is the maximum efficiency attained by the engine. We call η_{max} as the reversible efficiency which can be obtained naturally for any reversible heat engines

operating quasi statically and taking an infinite time to complete the process.

Let J_3 denotes the heat flux of the cold reservoir which is given by [22, 26] $J_3 \equiv \dot{Q}_c = \dot{Q}_h - \dot{W} = J_2 + J_1 X_1 T_c$. Using Eq.(1) one can obtain $X_1 = (J_1 - L_{12} X_2)/L_{11}$, then Eq.(2) and J_3 can be rewritten as

$$J_2 = \frac{L_{21}}{L_{11}} J_1 + L_{22}(1 - q^2) X_2 - r_h J_1^2, \quad (11)$$

$$J_3 = \frac{L_{21} T_c}{L_{11} T} J_1 + L_{22}(1 - q^2) X_2 + r_c J_1^2, \quad (12)$$

where $r_c = \frac{T_c}{L_{11}} - r_h$ and $q = \frac{L_{21}}{\sqrt{L_{11} L_{22}}}$ with $|q| \leq 1$ is the coefficient of the coupling strength [31]. Under the condition $|q| = 1$ called as the tight coupling, the second term $L_{22}(1 - q^2) X_2$ known as the heat leakage from the hot heat source to the cold heat reservoir vanishes [22, 26].

By using the above relations the entropy production rate, $\dot{S} = -\frac{J_2}{T_h} + \frac{J_3}{T_c}$ [26] can be written as [26, 28],

$$\dot{S} = L_{22}(1 - q^2) X_2^2 + \left\{ \frac{r_h}{T_h} + \frac{r_c}{T_c} \right\} J_1^2 \geq 0. \quad (13)$$

Since $r_h > 0$ and also the first term in the above equation is greater than or equal to zero, one can naturally make an assumption that $r_c > 0$ [26, 28] such that which should ensure the non negativity of the entropy production. In our study, we did not make such an assumption that the value of r_c should be greater than zero. However, in order to make the positive entropy production rate, we impose the condition $\left\{ \frac{r_h}{T_h} + \frac{r_c}{T_c} \right\} \geq 0$. Since $r_c = \frac{T_c}{L_{11}} - r_h$, this condition becomes, $\frac{1}{L_{11}} - r_h \left\{ \frac{1}{T_c} - \frac{1}{T_h} \right\} \geq 0$. For time varying hot heat source, the above condition can be rewritten as

$$X_2 L_{11} r_h \leq 1. \quad (14)$$

Under this condition, the entropy production rate becomes zero when $X_2 L_{11} r_h = 1$ and $|q| = 1$.

The rate of decrease of temperature T of the hot heat source when the heat engine operates from the initial temperature T_h to the final temperature T_c is given by [22, 24]

$$J_2 = -C_v \dot{T}. \quad (15)$$

The above equation also provides the relation that connects the temperature T to the time t with $\dot{T} = \frac{dT}{dt} \neq 0$ in general. Then Eq.(11) can be written as

$$r_h J_1^2 - \frac{L_{21}}{L_{11}} J_1 - L_{22}(1 - q^2) X_2 - C_v \dot{T} = 0. \quad (16)$$

Let us take $g = L_{22}(1 - q^2) X_2$ and $a_0 = \frac{L_{21}}{L_{11}}$, the above equation can then be written simply as

$$r_h J_1^2 - a_0 J_1 - g - C_v \dot{T} = 0. \quad (17)$$

In terms of g and a_0 , Eq.(12) can be written as

$$J_3 = \frac{T_c}{T} a_0 J_1 + g + \left(\frac{T_c}{L_{11}} - r_h \right) J_1^2. \quad (18)$$

Using Eq.(17) in the above equation for J_1^2 and after simplification one can get,

$$J_3 = a_0(\beta - X_2 T_c) J_1 + \beta g + (\beta - 1) C_v \dot{T}, \quad (19)$$

where $\beta = \frac{T_c}{L_{11} r_h} = \frac{r_c}{r_h} + 1$ [29]. Here r_c/r_h is the ratio of power dissipation between the cold and hot reservoirs. Using Eq.(14), $\beta = \frac{X_2 T_c}{X_2 L_{11} r_h}$ can takes value $\geq X_2 T_c$ and equality holds when $X_2 L_{11} r_h = 1$.

Since Eq.(17) is quadratic in J_1 , it has two roots J_1^+ and J_1^- which are given by

$$J_1^\pm = \frac{a_0}{2r_h} \left[1 \pm \sqrt{1 + \frac{4r_h}{a_0^2} (g + C_v \dot{T})} \right]. \quad (20)$$

$$= \frac{1}{a_0 a_1} \left[1 \pm \sqrt{1 + 2a_1 (g + C_v \dot{T})} \right]. \quad (21)$$

where $a_1 = 2r_h/a_0^2$. Using one of the values of J_1 , say $J_1 = J_1^+$, Eq.(19) can be expressed as a function of T and \dot{T} as

$$J_3(T, \dot{T}) = a_0(\beta - X_2 T_c) J_1^+ + \beta g + (\beta - 1) C_v \dot{T}.$$

The above equation can be rewritten as

$$J_3(T, \dot{T}) = k \left[1 + \sqrt{p} \right] + \beta g + (\beta - 1) C_v \dot{T}. \quad (22)$$

Here, we have taken $k = \frac{(\beta - X_2 T_c)}{a_1}$ and $p = 1 + 2a_1 (g + C_v \dot{T})$ for notational convenience. In our further calculation, we assume that β, a_0, a_1, C_v and the Onsager coefficients depends only on the temperature. Therefore, the leakage term g and k depends only on T but p depends on both T and \dot{T} . Thus,

$$k(T) = \frac{\beta(T) - X_2(T) T_c}{a_1(T)}$$

and

$$p(T, \dot{T}) = 1 + 2a_1(T) \left(g(T) + C_v(T) \dot{T} \right).$$

The partial differentiation of p with respect to T and \dot{T} is given by

$$\frac{\partial p}{\partial \dot{T}} = 2a_1 C_v \quad (23)$$

$$\frac{\partial p}{\partial T} = 2 \frac{\partial(a_1 g)}{\partial T} + 2 \dot{T} \frac{\partial(a_1 C_v)}{\partial T}. \quad (24)$$

By using Eqs.(22 - 24) one can calculate

$$\begin{aligned} \frac{\partial J_3}{\partial T} &= \frac{\partial}{\partial T} \left(k[1 + \sqrt{p}] + \beta g \right) \\ &+ \dot{T} \frac{\partial}{\partial T} \left((\beta - 1)C_v \right). \end{aligned} \quad (25)$$

$$\frac{\partial J_3}{\partial \dot{T}} = \frac{k}{\sqrt{p}} a_1 C_v + (\beta - 1)C_v. \quad (26)$$

$$\frac{\partial}{\partial \dot{T}} \left(\frac{\partial J_3}{\partial \dot{T}} \right) = -\frac{k}{p^{3/2}} a_1^2 C_v^2. \quad (27)$$

$$\begin{aligned} \frac{\partial}{\partial T} \left(\frac{\partial J_3}{\partial \dot{T}} \right) &= \frac{\partial}{\partial T} \left(\frac{k}{\sqrt{p}} a_1 C_v \right) \\ &+ \frac{\partial}{\partial T} \left((\beta - 1)C_v \right). \end{aligned} \quad (28)$$

III. THERMODYNAMIC OPTIMIZATION

The heat Q_h and the work output W can obtain from J_2 and J_3 in the time interval 0 to τ is [22]

$$Q_h = \int_0^\tau J_2(t) dt = - \int_{T_h}^{T_c} C_v dT = U_h - U_c, \quad (29)$$

$$W = \int_0^\tau \dot{W}(t) dt = U_h - U_c - \int_0^\tau J_3(t) dt, \quad (30)$$

where $\dot{W} = \frac{dW}{dt} = J_2 - J_3$. Hence, the total power P and the efficiency η can be obtained as [22]

$$P = \frac{W}{\tau} = \frac{U_h - U_c - \int_0^\tau J_3(t) dt}{\tau}, \quad (31)$$

$$\eta = \frac{W}{Q_h} = 1 - \frac{\int_0^\tau J_3(t) dt}{U_h - U_c}. \quad (32)$$

In order to maximize the work and hence obtain the maximum efficiency, we express $J_3(t)$ as a function of T and \dot{T} as in Eq.(22) and then minimize the integral $\int_0^\tau J_3(T, \dot{T}) dt$ in the above equation by solving the following Euler-Lagrange equation for $T(t)$ [22, 24]

$$\frac{d}{dt} \left(\frac{\partial J_3(T, \dot{T})}{\partial \dot{T}} \right) - \frac{\partial J_3(T, \dot{T})}{\partial T} = 0. \quad (33)$$

Eq.(33) can be rewritten in terms of $T(t)$ as

$$\ddot{T} \frac{\partial}{\partial \dot{T}} \left(\frac{\partial J_3}{\partial \dot{T}} \right) + \dot{T} \frac{\partial}{\partial T} \left(\frac{\partial J_3}{\partial \dot{T}} \right) - \frac{\partial J_3}{\partial T} = 0. \quad (34)$$

By using Eqs.(25 -28) in the above equation one can obtain

$$\begin{aligned} -\ddot{T} \frac{k a_1^2 C_v^2}{p^{3/2}} + \dot{T} \frac{\partial}{\partial T} \left(\frac{k}{\sqrt{p}} a_1 C_v \right) \\ - \frac{\partial}{\partial T} \left(k[1 + \sqrt{p}] + \beta g \right) = 0. \end{aligned} \quad (35)$$

Since

$$\frac{d}{dt} \left(\frac{k}{\sqrt{p}} a_1 C_v \right) = -\ddot{T} \frac{k a_1^2 C_v^2}{p^{3/2}} + \dot{T} \frac{\partial}{\partial T} \left(\frac{k}{\sqrt{p}} a_1 C_v \right) \quad (36)$$

and

$$\frac{\partial}{\partial \dot{T}} \left(k[1 + \sqrt{p}] + \beta g \right) = \frac{k}{\sqrt{p}} a_1 C_v, \quad (37)$$

Eq.(35) can be rewritten as

$$\frac{d}{dt} \left(\frac{\partial Y}{\partial \dot{T}} \right) - \frac{\partial Y}{\partial T} = 0, \quad (38)$$

where $Y(T, \dot{T}) = k[1 + \sqrt{p}] + \beta g$. We also get the same type of Eq.(38) for the other value of $J_1 = J_1^-$ with $Y(T, \dot{T}) = k[1 - \sqrt{p}] + \beta g$. Therefore for two different values of $J_1 = J_1^\pm$, one can get

$$\frac{d}{dt} \left(\frac{\partial Y}{\partial \dot{T}} \right) - \frac{\partial Y}{\partial T} = 0. \quad (39)$$

Multiply throughout by \dot{T} in the above equation, we obtain

$$\frac{d}{dt} \left(\dot{T} \frac{\partial Y}{\partial \dot{T}} - Y \right) = 0. \quad (40)$$

After integrating the above equation one can get

$$\dot{T} \frac{\partial Y(T, \dot{T})}{\partial \dot{T}} - Y(T, \dot{T}) = A, \quad (41)$$

where A is a τ dependent integration constant. As similar to Eq.(26) of Ref.[24], for any coupling strength $|q| \leq 1$, we have obtained the necessary condition to achieve an optimized work output from the minimally nonlinear irreversible model of heat engines. Since Eq.(41) can not be obtained in the linear regime [24], our study ensures that the minimally nonlinear irreversible model can be considered as a simplest and suitable model for studying heat engines in the nonlinear regime even though the dissipation term is present explicitly in this model as criticized earlier [30].

It should be noted that the condition in terms of $Y(T, \dot{T})$ obtained in our study is not for the entropy production rate as given in Ref.[24]. Eq.(41) is a highly nonlinear implicit differential equation [33], it may be difficult to simplify this equation for further analysis and

hence we do not try any other optimization [34] to minimize the integral $\int_0^\tau J_3(t)dt$ further. Under this optimization condition Eq.(22) becomes

$$\begin{aligned} J_3(T, \dot{T}) &= \dot{T} \frac{\partial Y(T, \dot{T})}{\partial \dot{T}} - A + (\beta - 1)C_v \dot{T} \\ &= \pm \frac{k}{\sqrt{p}} a_1 C_v \dot{T} - A + (\beta - 1)C_v \dot{T}. \end{aligned}$$

In the above equation, the symbol (\pm) indicates the results that are obtained for the two values of $J_1 = J_1^\pm$ with $Y(T, \dot{T}) = k[1 \pm \sqrt{p}] + \beta g$.

The condition for positive entropy production rate (Eq.14) can be written in terms of β as $X_2 L_{11} r_h = \frac{X_2 \dot{T}_c}{\beta} \leq 1$, then

$$\beta \geq X_2 T_c. \quad (42)$$

For the lowest value of $\beta = X_2 T_c$, $k = \frac{(\beta - X_2 T_c)}{a_1} = 0$ and hence from Eqs. (37) and (41) with $Y(T, \dot{T}) = k[1 \pm \sqrt{p}] + \beta g$ we get

$$\begin{aligned} A &= \pm \frac{k}{\sqrt{p}} a_1 C_v \dot{T} - \left(k[1 \pm \sqrt{p}] + \beta g \right) \\ &= -X_2 T_c g. \end{aligned} \quad (43)$$

Then, the optimized flux is given by

$$J_3(T, \dot{T}) = X_2 T_c g + (X_2 T_c - 1)C_v \dot{T}. \quad (44)$$

The above equation has been obtained by optimizing J_3 with $\beta = X_2 T_c$ for any value of the coupling strength $|q| \leq 1$. For this minimum value of β the entropy production should be independent of time under the tight coupling condition, $|q| = 1$. In this condition the leakage term $g = 0$ and the equation (44) becomes

$$J_3(T, \dot{T}) = (X_2 T_c - 1)C_v \dot{T} = -\frac{T_c}{T} C_v \dot{T}. \quad (45)$$

Integrating the above equation from 0 to τ and using Eq.(9), we get

$$\int_0^\tau J_3(T, \dot{T})dt = -T_c \int_{T_h}^{T_c} \frac{C_v}{T} dT = T_c(S_h - S_c).$$

By using Eq.(32), the optimized efficiency η can be obtained as

$$\begin{aligned} \eta &= 1 - \frac{\int_0^\tau J_3(T, \dot{T})dt}{U_h - U_c} \\ &= 1 - \frac{T_c(S_h - S_c)}{U_h - U_c} \\ &= \eta_{max}. \end{aligned} \quad (46)$$

Thus, the reversible efficiency has been obtained from the minimally nonlinear irreversible heat engine under the tight coupling condition. In the case of $C_v \rightarrow \infty$, for an isothermal environment η_{max} recovers the usual

Carnot efficiency η_C by the definition [22] $\frac{U_h - U_c}{T_h} = \frac{Q_h}{T_h} = S_h - S_c$. The maximum work (Exergy) extracted and also the total power obtained from this nonlinear irreversible heat engine are obtained as

$$W = U_h - U_c - T_c(S_h - S_c) \equiv E, \quad (47)$$

$$P = \frac{W}{\tau} = \frac{E}{\tau}. \quad (48)$$

Our result showed that the reversible efficiency obtained from the nonlinear irreversible heat engines under the tight coupling condition is not necessarily to be in the regime of maximum or zero power output. In what follows, we try to calculate the efficiency at maximum power from the minimally nonlinear irreversible heat engines under the non-tight coupling condition.

IV. EFFICIENCY AT MAXIMUM POWER UNDER THE NON-TIGHT COUPLING CONDITION

Under the non-tight coupling condition, $|q| \neq 1$ and hence the leakage term becomes non-zero ($g \neq 0$) for $L_{ij} > 0$. Since the integration constant $A = \beta g$ as obtained from Eq.(43) also depends on τ and $\beta = X_2 T_c$ is a function of T alone, one can expect g should also depend on τ for a given β . For the simplest choice, we take $g = B/(\beta \tau^2)$ where B is a constant and using this value of $g \neq 0$ in Eq.(44), we get

$$J_3(T, \dot{T}) = \frac{B}{\tau^2} + (X_2 T_c - 1)C_v \dot{T}. \quad (49)$$

Integrating the above equation from 0 to τ and using Eqs.(7),(9) and (31) we obtain

$$\int_0^\tau J_3(T, \dot{T})dt = \frac{B}{\tau^2} \int_0^\tau dt - T_c \int_{T_h}^{T_c} \frac{C_v}{T} dT.$$

$$\int_0^\tau J_3(T, \dot{T})dt = \frac{B}{\tau} + T_c(S_h - S_c). \quad (50)$$

and the total power

$$P = \frac{1}{\tau} \left(U_h - U_c - \frac{B}{\tau} - T_c(S_h - S_c) \right).$$

$$P = \frac{1}{\tau} \left(E - \frac{B}{\tau} \right). \quad (51)$$

It should be noted that the total power goes to zero in the quasi static limit $\tau \rightarrow \infty$. In order to find out the value of $\tau = \tau^*$ in which the total power is maximum, one can maximize Eq.(51) with respect to τ as

$$\frac{dP}{d\tau} = \frac{-E}{\tau^2} + \frac{2B}{\tau^3} = 0 \quad (52)$$

and obtain

$$\tau^* = \frac{2B}{E}. \quad (53)$$

With this value of τ^* , we obtain the maximum power

$$P^* = \frac{E^2}{4B}. \quad (54)$$

Using Eqs.(10) and (54), we obtain the work output and the efficiency at maximum power under the non-tight coupling condition as

$$W^* = P^* \tau^* = \frac{E}{2}, \quad (55)$$

$$\eta^* = \frac{W^*}{U_h - U_c} = \frac{1}{2} \eta_{max}. \quad (56)$$

The above result shows that the efficiency at maximum power is equal to half of the reversible efficiency and the corresponding maximum work is half the exergy. Our final result is exactly the same as the one obtained earlier for the study of exergy [22] in the case of linear irreversible heat engines under the tight coupling condition. This shows that the efficiency and the work at maximum power obtained from the linear irreversible heat engines under the tight coupling [22] is a special case of the efficiency at maximum power obtained from the minimally nonlinear irreversible heat engine under the non-tight coupling condition for a specific value of g .

V. CONCLUSION

Using the general formulation of the irreversible thermodynamics we studied the optimized work and the efficiency of minimally nonlinear irreversible heat engines operating between finite sized hot and infinite sized cold reservoirs. We obtained the necessary condition to

achieve an optimized work output. Our condition obtained in the case of minimally nonlinear irreversible model resembles with the one obtained recently [24] for the generalized study of the irreversible heat engines in the nonlinear regime. This condition can not be obtained in linear irreversible models [24]. Our result supports the fact that the minimally nonlinear irreversible model can be considered as a simplest and consistent model to study heat engines in the nonlinear regime.

We used the optimization condition, Eq.(41), and calculated the maximum work and efficiency of the minimally nonlinear irreversible heat engines. Earlier studies for the irreversible heat engines showed that the tight coupling condition serves as an upper bound of the efficiency at maximum power. Even though the universal feature in the efficiency at maximum power has been obtained for the linear irreversible heat engine, the reversible efficiency can not be reached at finite power even for the tight coupling condition which in general provides the upper bound of the efficiency at the maximum power. Interestingly, our result showed that the reversible efficiency can be achieved at finite power for nonlinear irreversible heat engine under the tight coupling condition. Our results also showed that the reversible efficiency obtained from the nonlinear irreversible heat engines in the tight coupling condition is not necessarily to be in the regime of maximum or zero power output.

We have calculated the efficiency at maximum power from the nonlinear irreversible heat engine under the non-tight coupling condition for a specific value of g and found that the efficiency at maximum power is equal to the half of the reversible efficiency and the corresponding maximum work is half the exergy. This result is exactly same as the efficiency and the work at maximum power obtained from the linear irreversible heat engines under the tight coupling condition [22].

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